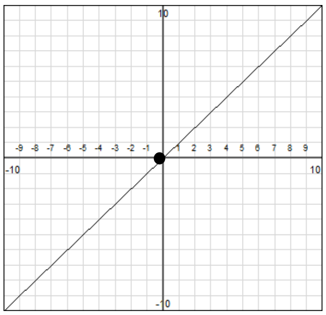
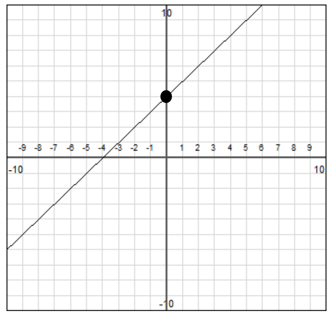
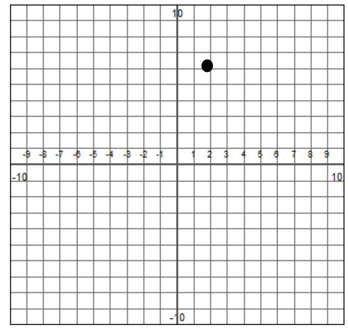
Standard Form of Conics

When we teach linear equations in Algebra1, we teach the simplest linear function (the mother function) as $y = x$. We then usually lead to the understanding of the effects of the slope and the y-intercept on the original. Let's focus on the point on the origin.

The shifting that can occur to the point that was on the origin $(0,0)$ is a combination of a horizontal shift and a vertical shift. We could write the equation of the second graph as $y-4 = x-0$ showing the vertical shift as +4 and the horizontal shift as 0. Instead, we write $y = x+4$.

What would the equation be of the linear function that passes through the point in the third graph with slope=1? If the point is the vertical shift in the origin of +6 and the horizontal shift of +2, we could write $y-6 = x-2$, which of course simplifies to However, if we did teach $y-k = m(x-h)$ form, then when students get to conics, they would probably have an easier time.

We could recognize $(y-6) = 1(x-2)$ as a linear equation because both variables are to degree one, and we could see that the point that is on the origin when in simplest form has now shifted to $(2,6)$. The movement from that point to the next is up 1 over 1 as determined by the slope.

Conics (Completing the square to find the standard form of conics):

The general form of a shifted conic is $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (both A and C ≠0).

A quick view

- If A or C = 0, then the conic is a parabola.
- If A and C have the same sign, then the conic is an ellipse. If A=C, then the conic is a circle.
- If A and C have opposite signs, then the conic is a hyperbola.

Transforming the equation from general form into standard form confirms which conic we have and makes graphing the conic easy. Transforming the equation requires completing the square.

Refer to the Completing the Square Lesson on the ACOE website under General Resources.

Parabolas:

Taking a step back to algebra1, we recognize $y = ax^2$ a parabola that opens up or down with vertex $(0,0)$ and axis of symmetry $x = 0$. We did not define the parabola beyond being "u-shaped". Now we are going to define a parabola with a little more detail. A parabola is the shape produced when you consider all the points equidistant from a line (called the directrix) and a point not on the line (called the focus). Parabolas in the form $x^2 = 4py$ open up or down, parabolas in the form $y^2 = 4px$ open side to side.

Example 1: $x^2 - 10x + 16y = -57$

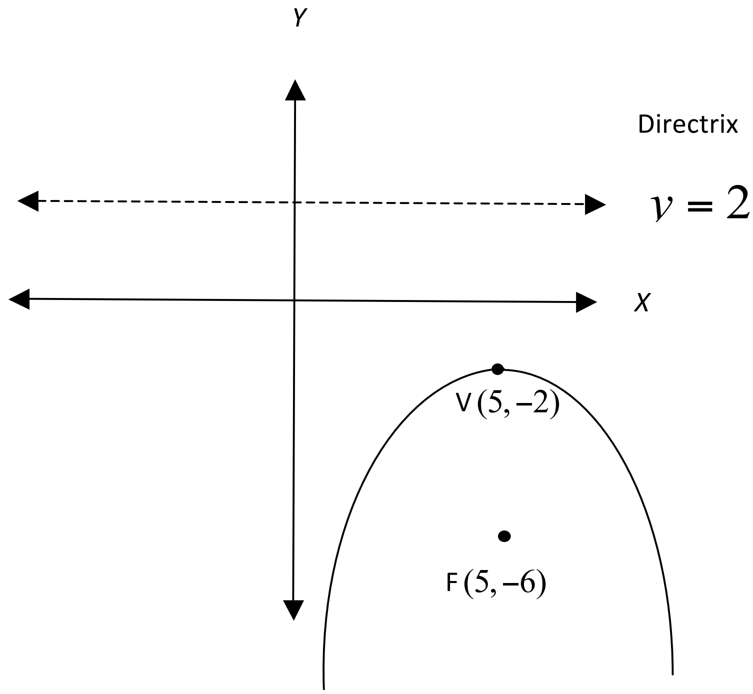
$x^2 - 10x = -16y - 57$ We isolate the squared variable with its linear term

$x^2 - 10x + 25 = -16y - 57 + 25$ Add 25 to both sides to complete the square

$(x-5)^2 = -16y - 32$ Factor

$(x-5)^2 = -16(y+2)$ Write in factored form (Verify $C = 0$ so a parabola)

Graph as before with $V(5,-2)$ $\quad \begin{aligned} 4p &= -16 \\ p &= -4 \end{aligned}$ so it opens down



<u>Example 2</u>: $x^2 - 2y^2 - 6x - 4y + 3 = 0$

$$x^2 - 6x \quad - 2y^2 - 4y \quad = -3$$

$$x^2 - 6x \quad - 2\left(y^2 + 2y \quad\right) = -3$$

$$(x^2 - 6x + 9) - 2(y^2 + 2y + 1) = -3 + 9 - 2$$

$$(x-3)^2 - 2(y+1)^2 = 4$$

$$2$$

$$\frac{(x-3)^2}{4} - \frac{2(y+1)^2}{4} = \frac{4}{4}$$

$$\frac{\left(x-3\right)^2}{4} - \frac{(y+1)^2}{2} = 1$$

Have students work in pairs taking turns to explain what has happened in each step.

Why did we add -2 to the right side instead of 1? *(because we need to distribute the -2)*

What kind of shape is this? How do we know? *(Hyperbola. A and C had different signs. When transformed, the equation is set equal to 1and we are subtracting.)*

You try #1: Transform into standard form and determine the conic:

$$2x^2 + y^2 + 4x + 2y - 6 = 0$$

$$2x^2 + 4x \quad + y^2 + 2y \quad = 6$$
$$2(x^2 + 2x + 1) + (y^2 + 2y + 1) = 6 + 2 + 1$$
$$2(x+1)^2 + \left(y+1\right)^2 = 9$$
$$\frac{2(x+1)^2}{9} + \frac{\left(y+1\right)^2}{9} = 1$$
$$\frac{(x+1)^2}{\frac{9}{2}} + \frac{\left(y+1\right)^2}{9} = 1$$

Note: The 1st term is rewritten in the form $\frac{(x+1)^2}{b^2}$

Have students work in pairs taking turns to explain what has happened in each step.

Why did we add 2 to the right side instead of 1? *(because we need to distribute the 2)*

What kind of shape is this? How do we know? *(Ellipse. A and C have the same signs. When transformed, the equation is set equal to 1and we are adding with unlike denominators.)*

You try #2: Transform into standard form and determine the conic:

$$4x^2 + 9y^2 - 8x - 54y + 49 = 0$$

$$4x^2 - 8x \quad + 9y^2 - 54y = -49$$
$$4(x^2 - 2x + 1) + 9(y^2 - 6y + 9) = -49 + 4 + 81$$
$$4(x-1)^2 + 9\left(y-3\right)^2 = 36$$
$$\frac{4(x-1)^2}{36} + \frac{9\left(y-3\right)^2}{36} = \frac{36}{36}$$
$$\frac{(x-1)^2}{\frac{9}{2}} + \frac{\left(y-3\right)^2}{4} = 1$$

Have students work in pairs taking turns to explain what has happened in each step.

Why did we add 4 and 81 to the right side instead of 1 and 9? *(because we need to distribute the 4 and the 9)*

What kind of shape is this? How do we know? *(Ellipse. A and C have the same signs. When transformed, the equation is set equal to 1 and we are adding with unlike denominators.)*

Example 3: Transform into standard form and determine the conic:

$$x^2 + 3y^2 - 4x + 6y + 7 = 0$$

$$x^2 - 4x \quad + 3y^2 + 6y = -7$$
$$(x^2 - 4x + 4) + 3(y^2 + 2y + 1) = -7 + 4 + 3$$
$$(x-2)^2 + 3\left(y+1\right)^2 = 0$$

The result of zero means that there is only one point on this graph. (Both $(x-2)^2$ and $3(y+1)^2$ are positive.)

Only $(2,-1)$ satisfies this equation.

If we obtain a negative on the right side, then there would be no solution, so no graph. This is called degenerate.

<u>You try #3</u>: Transform into standard form and determine the conic:

$$4x^2 + y^2 - 8x + 2y + 8 = 0$$

$$4x^2 - 8x \quad + y^2 + 2y = -8$$
$$4(x^2 - 2x + 1) + (y^2 + 2y + 1) = -8 + 4 + 1$$
$$4(x-1)^2 + \left(y-1\right)^2 = -3$$

This is degenerate so therefore has no solution. The sum of two squares will never be negative.

<u>Example 4:</u> Transform into standard form and determine the conic:

$$9x^2 - y^2 + 18x + 4y + 5 = 0$$

$$9(x^2 + 2x + 1) - (y^2 - 4y + 4) = -5 + 9 - 4$$
$$9(x+1)^2 - \left(y-2\right)^2 = 0$$

Degenerate, however since it is subtraction, we can attempt to solve.

$$9\left(x+1\right)^2 = \left(y-2\right)^2$$
$$\pm\sqrt{9\left(x+1\right)^2} = \pm\sqrt{\left(y-2\right)^2}$$
$$\pm 3(x+1) = y - 2$$
$$y = 2 \pm 3(x+1)$$

$$y = 2 \pm 3(x+1)$$
$$y = 2 + 3(x+1) \quad \text{or} \quad y = 2 - 3(x+1)$$
$$y = 2 + 3x + 3 \quad \text{or} \quad y = 2 - 3x - 3$$
$$y = 3x + 5 \quad \text{or} \quad y = -3x - 1$$

This examples graphs out to be two lines.